(ϵ_2) for Mg_2Pb. It is to be hoped that the publishing of this band structure plus the resolution of the crystal preparation problem[6] will stimulate the measurement of optical properties of Mg_2Pb like the reflectivity[18] and modulated reflection[19] work that has been done on the other Mg_2X compounds.

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Simultaneous Generation of Transition Radiation and Bremsstrahlung from a Thin Foil. I*

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The usual procedure of reducing transition-radiation data by simply subtracting bremsstrahlung yield from the total yield ignores the coherent interference effect between transition radiation and bremsstrahlung. Assuming single scattering, we have analyzed the simultaneous generation of transition radiation and bremsstrahlung of a charged particle normally incident on a thin slab. The results obtained here not only should be useful for reducing experimental data on transition radiation, but also should provide a new independent means to deduce the mean-square angle for single scattering.

I. INTRODUCTION

When a charged particle passes through a thin dielectric slab without deflection along the direction of the normal to the plane surfaces of the slab, transition radiation with its characteristic polarization will be emitted.[1] Transition radiation from a normally incident charged particle is always polarized in the plane of emission determined by the direction of observation and the normal to the slab. However, in practice charged particles may suffer scattering in the slab, in which case bremsstrahlung is also generated. The usual procedure of reducing transition-radiation data by simply subtracting bremsstrahlung yield from the total yield ignores the coherent interference effect between transition radiation and bremsstrahlung. The simultaneous generation of transition radiation and bremsstrahlung from charged particles incident on a semiinfinite medium was calculated, taking into account multiple scattering of charged particles inside the semiinfinite medium.[2,3] In many practical cases very thin foils are used, for which situation a single-scattering theory may be appropriate.

Assuming single scattering, we have calculated the simultaneous generation of transition radiation and bremsstrahlung from a charged particle normally incident on a thin parallel plane dielectric slab containing randomly distributed scattering centers. First, the transition radiation and bremsstrahlung generated in a particular trajectory of a charged particle, which is scattered through an angle Θ at a depth ζ inside the slab, is calculated and then averaged over the scattering angle Θ and the depth ζ. For fast-charged particles used in transition-radiation experiments small angle scattering may be assumed and consistent expansions in Θ up to and including Θ^2 are made in the calculation. The results obtained here should be useful not only for reducing experimental data, new or old, on transition radiation, but also should provide a new independent means to deduce the mean-square angle $\langle\Theta^2\rangle$ for single scattering of energetic-charged particles in a given material.

In Sec. II the current density corresponding to a particular trajectory of a charged particle scattered through an angle Θ at a depth ζ is given and the inhomogeneous solution of the Hertz vector due to this current density is calculated. The coefficients of the homogeneous solutions of the Hertz vectors, which correspond to transition radiation, are determined in Sec. III from the boundary conditions at the plane boundaries of the slab. In Sec. IV applying the saddle-point method in the far radiation zone, the Poynting flux is evaluated for both polarizations parallel (∥) and perpendicular (⊥) to the plane of emission, and for both forward and backward directions. The results of the averaging over the scattering angle Θ and the scattering depth ζ are also presented.

II. CURRENT DENSITY AND INHOMOGENEOUS SOLUTION

Consider a uniformly moving charged particle normally incident on a thin slab of thickness d (Fig. 1) and assume that the particle undergoes single scattering at a depth $z=\zeta$ by deviating through a small angle Θ from the original direction without changing the magnitude of its velocity v. The

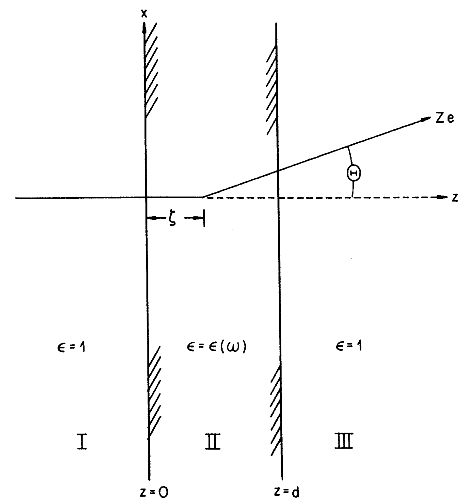

FIG. 1. Geometry for single scattering of the electron at the depth ζ inside the thin foil which has plane surfaces parallel to the xy plane and extends from $z=0$ to $z=d$.

particle carries a charge Ze and the original direction of motion is taken as the z axis. The two plane surfaces of the slab, situated at $z=0$ and $z=d$, divide space into regions I, II, and III. In the empty spaces I and II the dielectric constant is equal to 1, and inside the slab II the frequency-dependent complex dielectric constant is denoted by $\epsilon(\omega)$. If the charged particle emerges with velocity parallel with xz plane, then the y components of the current and the Hertz vector are both zero and will not appear in the following calculation. The remaining components of the current density $\vec{j}(\vec{r},t)$ are given by

$$\begin{Bmatrix} j_x \\ j_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} Zev\delta(x)\delta(y)\delta(z-vt), \quad \text{for} \quad z \leq \zeta \tag{1a}$$

and

$$\begin{Bmatrix} j_x \\ j_z \end{Bmatrix} = \begin{Bmatrix} \Theta \\ 1-\frac{1}{2}\Theta^2 \end{Bmatrix} Zev\delta(x-(vt-\zeta)\Theta) \times\delta(y)\delta(z-vt+(vt-\zeta)\tfrac{1}{2}\Theta^2), \quad \text{for} \quad z \geq \zeta\,. \tag{1b}$$

Here we have assumed that the particle crosses the front surface $z=0$ of the slab at time $t=0$, and $\delta(x)$ denotes Dirac's δ function. Terms of higher order than Θ^2 are neglected.

The Fourier transforms of Eqs. (1)

$$\vec{J}(k_x,k_y,\omega\,|\,z)=1/(2\pi)^{3/2}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}dt$$

$$\times\ \vec{j}(\vec{r},t)\exp[-i(xk_x+yk_y-\omega t)]\ ,$$

can be found easily:

$$\begin{Bmatrix}J_x\\J_z\end{Bmatrix}=\begin{Bmatrix}0\\1\end{Bmatrix}\frac{Ze}{(2\pi)^{3/2}}\,e^{i\eta z},\quad \text{for}\quad z\le\zeta\ , \tag{2a}$$

$$\begin{Bmatrix}J_x\\J_z\end{Bmatrix}=\begin{Bmatrix}\Theta\\1-\frac{1}{2}\Theta^2\end{Bmatrix}\frac{Ze}{(2\pi)^{3/2}}\,e^{-i\xi\zeta}e^{i\gamma z},\quad \text{for}\quad z\ge\zeta\ , \tag{2b}$$

where the abreviations $\eta=\omega/v$, $\xi=-k_x\Theta+(\frac{1}{2}\eta)\Theta^2$, and $\gamma=\eta+\xi$ were used.

Starting from the well-known relation between the vector potential $\vec{A}(\vec{r},t)$ of the electromagnetic field and the current density $\vec{j}(\vec{r},t)$

$$\left(\nabla^2-\frac{\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}=-\frac{4\pi}{c}\,\vec{j}\quad ,$$

we may introduce the Hertz vector $\vec{\Pi}(\vec{r},t)$ via

$$\vec{A}(\vec{r},t)=\frac{\epsilon}{c}\frac{\partial\vec{\Pi}}{\partial t}$$

and its Fourier transform

$$\vec{\Pi}(k_x,k_y,\omega\,|\,z)=1/(2\pi)^{3/2}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}dt\ \times\vec{\Pi}(\vec{r},t)\exp[-i(xk_x+yk_y-\omega t)]\,.$$

Then we obtain the following equation relating the Fourier transforms of the Hertz vector and the current density:

$$\left(\frac{d^2}{dz^2}+k_z^2\right)\vec{\Pi}(k_x,k_y,\omega\,|\,z)=\frac{4\pi}{i\omega\epsilon}\,\vec{J}(k_x,k_y,\omega\,|\,z)\quad , \tag{3}$$

where $k_z^2=\epsilon(\omega)\omega^2/c^2-k_x^2-k_y^2$. We shall use the abbreviations $\kappa^2=k_x^2+k_y^2$ and

$$k_z=+(\epsilon\omega^2/c^2-\kappa^2)^{1/2}\quad .$$

The inhomogeneous solution to (3) is given by

$$\vec{\Pi}(z)=(1/2ik_z)\Big[e^{ik_zz}\int_{-\infty}^{z}e^{-ik_zz'}\vec{J}(z')\,dz' + e^{-ik_zz}\int_{z}^{\infty}e^{ik_zz'}\,\vec{J}(z')dz'\Big]\quad , \tag{4}$$

if $\mathrm{Im}(k_z)>0$. We shall denote the inhomogeneous solutions in the three regions I, II, and III (Fig. 1) by $\vec{\pi}_1(z)$, $\vec{\pi}_2(z)$, and $\vec{\pi}_3(z)$, respectively. The derivatives of these inhomogeneous solutions with respect to z will be designated by adding a prime to the respective solutions $\vec{\pi}_1'(z)$, $\vec{\pi}_2'(z)$, and $\vec{\pi}_3'(z)$. The value of $k_z=(\epsilon\omega^2/c^2-\kappa^2)^{1/2}$ depends on $\epsilon(\omega)$ and because of its frequent appearance below, we shall use the notations

$$q\equiv k_z=(\omega^2/c^2-\kappa^2)^{1/2}\quad ,$$

in regions I and III with $\epsilon=1$,

$$q'\equiv k_z=(\epsilon\omega^2/c^2-\kappa^2)^{1/2}\quad ,$$

in region II with $\epsilon(\omega)\ne1$. We need the following values of the inhomogeneous solutions and their derivatives at $z=0$ and $z=d$:

$$\begin{aligned}
&\vec{\pi}_1(0)=\begin{pmatrix}0\\1\end{pmatrix}f,\quad \frac{\vec{\pi}_1'(0)}{iq}=\begin{pmatrix}0\\1\end{pmatrix}f\Delta\ ,\\
&\vec{\pi}_2(0)=\begin{pmatrix}0\\1\end{pmatrix}\frac{f'}{\epsilon},\quad \frac{\vec{\pi}_2'(0)}{iq}=\begin{pmatrix}0\\1\end{pmatrix}\frac{f'}{\epsilon}\,\Delta\ ,\\
&\vec{\pi}_2(d)=\begin{pmatrix}\Theta\\1-\frac{1}{2}\Theta^2\end{pmatrix}\frac{g'}{\epsilon},\quad \frac{\vec{\pi}_2'(d)}{iq}=\begin{pmatrix}\Theta\\1-\frac{1}{2}\Theta^2\end{pmatrix}\frac{g'}{\epsilon}\,\frac{\gamma}{q}\ ,\\
&\vec{\pi}_3(d)=\begin{pmatrix}\Theta\\1-\frac{1}{2}\Theta^2\end{pmatrix}g\ ,\quad \frac{\vec{\pi}_3'(d)}{iq}=\begin{pmatrix}\Theta\\1-\frac{1}{2}\Theta^2\end{pmatrix}g\,\frac{\gamma}{q}\ ,
\end{aligned} \tag{5}$$

where the following abbreviations were used:

$$f(\text{or }f')=\frac{1}{q^2(\text{or }q'^2)-\eta}\ ,$$

$$g(\text{or }g')=\frac{e^{i\eta d}\,e^{i\xi(d-\zeta)}}{q^2(\text{or }q'^2)-\gamma^2}\ ,\quad\text{and}\quad \Delta=\frac{\eta}{q}\ .$$

In Eqs. (5) the common factor $(Ze/(2\pi)^{3/2})(4\pi/i\omega)$ has been omitted and will be restored in Sec. IV.

Before we can use the inhomogeneous solutions (5) as input data to determine the amplitudes of the free field [homogeneous solutions of Eq. (3)], these solutions must first be decomposed into terms of orders 1, Θ, and Θ^2. This involves a rather complicated algebra and results in lengthy intermediate expressions at this stage. After averaging over Θ and ζ, these expressions can be somewhat simplified; we shall present only the final results in Sec. IV.

III. BOUNDARY CONDITIONS AND DETERMINATION OF AMPLITUDES OF FREE FIELDS

The *total* Hertz vectors in regions I, II, and III will be represented by $\vec{\Pi}_1$, $\vec{\Pi}_2$, and $\vec{\Pi}_3$, respectively. Each of these Hertz vectors is the sum of a bound field [inhomogeneous solution of Eq. (3)] and a free field [homogeneous solution of Eq. (3)]. The inhomogeneous part was already calculated in Secs. I and II and we have the following total Hertz vectors in the three regions:

$$\begin{aligned}
&\Pi_{1x}(z)=Ae^{-iqz}+\pi_{1x}(z)\ ,\\
&\Pi_{1z}(z)=Be^{-iqz}+\pi_{1z}(z)\ ;\\
&\Pi_{2x}(z)=Ce^{iq'z}+De^{-iq'z}+\pi_{2x}(z)\ ,\\
&\Pi_{2z}(z)=Ee^{iq'z}+Fe^{-iq'z}+\pi_{2z}(z)\ ;\\
&\Pi_{3x}(z)=Ge^{iqz}+\pi_{3x}(z)\ ,\\
&\Pi_{3z}(z)=He^{iqz}+\pi_{3z}(z)\ .
\end{aligned} \tag{6}$$

The signs ± in the exponent $e^{i(\pm qz-\omega t)}$ were chosen according to the direction of propagation of the free waves. The coefficients (A,B) and (G,H) will be determined from the boundary conditions which require the continuity of the tangential components of both the total electric $\vec{E}$ and magnetic fields $\vec{H}$ at the interfaces $z=0$ and $z=d$. The boundary conditions

$$\epsilon\Pi_x=\epsilon'\Pi_x' \quad \text{and} \quad (d\Pi_x/dz)\epsilon=(d\Pi_x'/dz)\epsilon'$$

at $z=0$ and $z=d$ yield

$$\begin{aligned}
&A+\pi_{1x}(0)=\epsilon[C+D+\pi_{2x}(0)] \ , \\
&-iqA+\pi_{1x}'(0)=\epsilon[iq'C-iq'D+\pi_{2x}'(0)] \ ; \\
&\epsilon[Ce^{i\alpha}+De^{-i\alpha}+\pi_{2x}(d)]=Ge^{iqd}+\pi_{3x}(d) \ , \\
&\epsilon[iq'Ce^{i\alpha}-iq'De^{-i\alpha}+\pi_{2x}'(d)]=iqGe^{iqd}+\pi_{3x}'(d) \ .
\end{aligned} \tag{7}$$

Here we introduced the abbreviation $\alpha=q'd$. Solving the above equations, we obtain

$$A=[(U_1\mu-U_1'\mu')+(U_2-U_2')]/(\mu+\mu') \ , \tag{8a}$$

$$Ge^{iqd}=[-(U_1+U_1')-(U_2\mu+U_2'\mu')]/(\mu+\mu'), \tag{8b}$$

in which $\mu=\cos\alpha-i\delta\sin\alpha$, $\mu'=\cos\alpha-(i/\delta)\sin\alpha$, $\delta=q'/q$, and

$$\begin{pmatrix} U_1 \\ U_1' \end{pmatrix}=\begin{pmatrix} \epsilon\pi_{2x}(0)-\pi_{1x}(0) \\ [\epsilon\pi_{2x}'(0)-\pi_{1x}'(0)]/iq \end{pmatrix}$$

and (9)

$$\begin{pmatrix} U_2 \\ U_2' \end{pmatrix}=\begin{pmatrix} \pi_{3x}(d)-\epsilon\pi_{2x}(d) \\ [\pi_{3x}'(d)-\epsilon\pi_{2x}'(d)]/iq \end{pmatrix}.$$

From the other set of boundary conditions

$$\epsilon\Pi_z=\epsilon'\Pi_z' \quad \text{and} \quad \frac{d\Pi_x}{dx}+\frac{d\Pi_z}{dz}=\frac{d\Pi_x'}{dx}+\frac{d\Pi_z'}{dz}$$

at $z=0$ and $z=d$, we have

$$\begin{aligned}
&B+\pi_{1z}(0)=\epsilon[E+F+\pi_{2z}(0)] \ , \\
&-iqB+\pi_{1z}'(0)+(d\Pi_{1x}/dx)|_{z=0}=iq'(E-F) \\
&\qquad +\pi_{2z}'(0)+d\Pi_{2x}/dx|_{z=0} \quad ; \\
&\epsilon[Ee^{i\alpha}+Fe^{-i\alpha}+\pi_{2z}(d)]=He^{iqd}+\pi_{3z}(d) \ , \\
&iq'Ee^{i\alpha}-iq'Fe^{-i\alpha}+\pi_{2z}'(d)+(d\Pi_{2x}/dx)|_{z=d} \\
&\qquad =iqHe^{iqd}+\pi_{3z}'(d)+(d\Pi_{3x}/dx)|_{z=d} \ .
\end{aligned} \tag{10}$$

It should be noted that in the above equations the derivatives with respect to z are denoted by primes, while the terms with the derivatives with respect to x still involve the *total* fields, which include both bound and free fields. Solving the above equations, we get

$$B=[(V_1\nu-V_1'\nu')+(V_2-V_2')]/(\nu+\nu') \ , \tag{11a}$$

$$He^{iqd}=-[(V_1+V_1')+(V_2\nu+V_2'\nu')]/(\nu+\nu') \ , \tag{11b}$$

in which $\nu=\cos\alpha-i(\delta/\epsilon)\sin\alpha$ and $\nu'=\cos\alpha-i(\epsilon/\delta)\sin\alpha$. The quantities V_1, V_1', V_2, and V_2' are defined as

$$\begin{pmatrix} V_1 \\ V_1' \end{pmatrix}=\begin{pmatrix} \epsilon\pi_{2z}(0)-\pi_{1z}(0) \\ \dfrac{\pi_{2z}'(0)-\pi_{1z}'(0)}{iq} \end{pmatrix}+\begin{pmatrix} 0 \\ \dfrac{k_x}{q}\left(\dfrac{1}{\epsilon}-1\right)\Pi_{1x}(0) \end{pmatrix} ,$$

$$\begin{pmatrix} V_2 \\ V_2' \end{pmatrix}=\begin{pmatrix} \pi_{3z}(d)-\epsilon\pi_{2z}(d) \\ \dfrac{\pi_{3z}'(d)-\pi_{2z}'(d)}{iq} \end{pmatrix}+\begin{pmatrix} 0 \\ \dfrac{k_x}{q}\left(1-\dfrac{1}{\epsilon}\right)\Pi_{3x}(d) \end{pmatrix} . \tag{12}$$

Here we should note that $\Pi_{1x}(0)$ and $\Pi_{3x}(d)$ represent the total sum of the x components of free and bound fields which were determined from Eqs. (8) and (9).

By substituting the inhomogeneous solutions of various orders obtained in Sec. II into Eqs. (8), (9), (11), and (12), we obtain the corresponding amplitudes of the free fields:

$$\left(\frac{\omega}{c}\right)^2\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}+\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}\Theta+\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}\Theta^2 \ , \tag{13a}$$

$$\left(\frac{\omega}{c}\right)^2\begin{pmatrix} G \\ H \end{pmatrix}=\begin{pmatrix} G_0 \\ H_0 \end{pmatrix}+\begin{pmatrix} G_1 \\ H_1 \end{pmatrix}\Theta+\begin{pmatrix} G_2 \\ H_2 \end{pmatrix}\Theta^2 \ . \tag{13b}$$

Here we multiplied the left-hand sides of both equations by $(\omega/c)^2$ so that the quantities A_i, B_i, G_i, and H_i, which will be used in the final formulas, become dimensionless. We shall give explicit expressions for these quantities after averaging over all possible configurations in Sec. IV. Here we only note that $A_0=G_0=0$ and that from B_1, H_1, A_2, and G_2 one can factor out k_x. These observations will be useful below.

IV. FORWARD AND BACKWARD YIELDS

In order to find the Poynting flux at large distances we apply the saddle-point method to the Hertz vector:

$$\vec{\Pi}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}dk_x\int_{-\infty}^{+\infty}dk_y\int_{-\infty}^{+\infty}d\omega\,\frac{Ze}{(2\pi)^{3/2}}\frac{4\pi}{i\omega}\begin{pmatrix} G \\ H \end{pmatrix} \times\exp[i(xk_x+yk_y+zq-\omega t)] \ , \tag{14}$$

and obtain for large positive values of z

$$\vec{\Pi}(\vec{r},t)=-\frac{Ze\cos\theta}{\pi c}\int_{-\infty}^{+\infty}d\omega\begin{pmatrix} G \\ H \end{pmatrix}\frac{e^{i(kr-\omega t)}}{r} \ ,$$

where $\vec{r}=(r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)$ and $k=|\vec{k}|$. Furthermore, in the expressions for G and H one puts $q=(\omega/c)\cos\theta$, $q'=(\omega/c)(\epsilon-\sin^2\theta)^{1/2}$, and $k_x=(\omega/c)\sin\theta\cos\varphi$. In (14) we have restored the factor $[Ze/(2\pi)^{3/2}]4\pi/i\omega$ which has been omitted following Eqs. (5). Since $\vec{E}=(\omega^2/c^2)\vec{\Pi}$, one gets, in polar coordinates,

$$E_\theta=H_\varphi=-(Ze\cos\theta/\pi c)\int_{-\infty}^{+\infty}d\omega(\omega/c)^2$$

$$\times(e^{i(kr-\omega t)}/r)(G\cos\theta\cos\varphi - H\sin\theta) \ ,$$

$$E_\varphi = -H_\theta = -\frac{Ze\cos\theta}{\pi c}\int_{-\infty}^{+\infty} d\omega\left(\frac{\omega}{c}\right)^2 \frac{e^{i(kr-\omega t)}}{r}(-G\sin\varphi) \ .$$

The Poynting flux of photons polarized in the plane of emission and emitted in the forward hemisphere is given by

$$S_\parallel = \frac{c}{4\pi}\int_{-\infty}^{+\infty} E_\theta H_\varphi \, dt r^2 d\Omega$$

$$= d\Omega\,\frac{(Ze\cos\theta)^2}{\pi^2 c}\int_0^\infty d\omega\left(\frac{\omega}{c}\right)^4$$

$$\times |G\cos\theta\cos\varphi - H\sin\theta|^2 \ .$$

Finally we obtain for the number of photons emitted per unit frequency interval at frequency ω and per unit solid angle in the direction of θ

$$d^2N_\parallel/d\omega d\Omega = \Gamma(\omega/c)^4 |G\cos\theta\cos\varphi - H\sin\theta|^2 \ , \quad (15)$$

with $\Gamma = Z^2\alpha\cos^2\theta/\pi^2\omega$ and $\alpha = e^2/\hbar c$. Similarly we get

$$d^2N_\perp/d\omega d\Omega = \Gamma(\omega/c)^4 |G|^2\sin^2\varphi \ . \quad (16)$$

The corresponding formulas in the backward hemisphere are given by

$$d^2N_\parallel/d\omega d\Omega = \Gamma(\omega/c)^4 |A|\cos\theta|\cos\varphi + B\sin\theta|^2 , \quad (17)$$

$$d^2N_\perp/d\omega d\Omega = \Gamma(\omega/c)^4 |A|^2\sin^2\varphi \ . \quad (18)$$

Equations (15)–(18) must be averaged over Θ and ζ. Substituting (13b) into (15) and noting the observations made there on G_2 and H_1, we get

$$(\omega/c)^4 |G\cos\theta\cos\varphi - H\sin\theta|^2$$
$$= |(G_1\Theta + G_2\sin\theta\cos\varphi\,\Theta^2)\cos\theta\cos\varphi$$
$$-(H_0 + H_1\sin\theta\cos\varphi\,\Theta + H_2\Theta^2)\sin\theta|^2 \ .$$

Expanding the absolute square and taking averages over φ and Θ, the above expression becomes

$$|H_0|^2\sin^2\theta + [(H_0H_2^* + H_0^*H_2)\sin^2\theta$$
$$+\tfrac{1}{2}(H_0G_2^* + H_0^*G_2)\sin^2\theta\cos\theta$$
$$+\tfrac{1}{2}|G_1\cos\theta - H_1\sin^2\theta|^2]\langle\Theta^2\rangle \ .$$

It should be noted that H_0 and G_1 do not contain k_x and that H_1 and G_2, as already noted above, contain k_x only as an over-all factor which was explicitly taken out in the above averaging over φ. But H_2 still contains k_x^2 and one has to substitute the value $\frac{1}{2}$ for $\cos^2\varphi$ contained in H_2 through k_x^2.

Similar averaging over φ and Θ can be made for Eqs. (16)–(18) and the results of these averagings are included in the final results given below.

For the averaging over the scattering depth ζ, we first note that B_0, H_0, A_1, and G_1 do not contain ζ while all other amplitudes depend on ζ. In the absence of scattering, $|H_0|^2$ and $|B_0|^2$ contribute to the generation of transition radiation in the forward and backward hemispheres, respectively. Assuming uniform scattering probability per unit length inside the slab and taking the average over ζ from 0 to d, we obtain the final results.

Foward yield:

$$d^2N_\parallel/d\omega d\Omega = \Gamma(|H_0|^2\bar{\nu}^2\sin^2\theta + M\langle\Theta^2\rangle) \ ,$$

with

$$M = (H_0\langle H_2^*\rangle + H_0^*\langle H_2\rangle)\bar{\nu}^2\sin^2\theta$$
$$+\tfrac{1}{2}(H_0\langle G_2^*\rangle + H_0^*\langle G_2\rangle)\bar{\mu}\bar{\nu}\sin^2\theta\cos\theta$$
$$+\tfrac{1}{2}|G_1|^2\bar{\mu}^2\cos^2\theta + \tfrac{1}{2}\langle|H_1|^2\rangle\bar{\nu}^2\sin^4\theta$$
$$-\tfrac{1}{2}(G_1\langle H_1^*\rangle + G_1^*\langle H_1\rangle)\bar{\mu}\bar{\nu}\cos\theta\sin^2\theta \ ,$$

$$d^2N_\perp/d\omega d\Omega = \Gamma\,\tfrac{1}{2}|G_1|^2\bar{\mu}^2\langle\Theta^2\rangle \ ;$$

Backward yield:

$$d^2N_\parallel/d\omega d\Omega = \Gamma(|B_0|^2\bar{\nu}^2\sin^2\theta + M'\langle\Theta^2\rangle) \ ,$$

with

$$M' = (B_0\langle B_2^*\rangle + B_0^*\langle B_2\rangle)\bar{\nu}^2\sin^2\theta$$
$$+\tfrac{1}{2}(B_0\langle A_2^*\rangle + B_0^*\langle A_2\rangle)\bar{\mu}\bar{\nu}\sin^2\theta\cos\theta$$
$$+\tfrac{1}{2}|A_1|^2\bar{\mu}^2\cos^2\theta + \tfrac{1}{2}\langle|B_1|^2\rangle\bar{\nu}^2\sin^4\theta$$
$$+\tfrac{1}{2}(A_1\langle B_1^*\rangle + A_1^*\langle B_1\rangle)\bar{\mu}\bar{\nu}\cos\theta\sin^2\theta \ ,$$

$$d^2N_\perp/d\omega d\Omega = \Gamma\,\tfrac{1}{2}|A_1|^2\bar{\mu}^2\langle\Theta^2\rangle \ .$$

In these final equations we have taken out the common factor $\bar{\mu} = |\mu + \mu'|^{-1}$ out of A_i and G_i and, similarly, $\bar{\nu} = |\nu + \nu'|^{-1}$ out of B_i and H_i. The quantities which contain ζ and were averaged over ζ are indicated by angular brackets. Finally we list the explicit expressions for all amplitudes after being averaged over φ, Θ, and ζ:

$$B_0 = (P_1\nu - P_2\nu')P_0 - P_- \ ,$$
$$H_0 = (P_1 + P_2)P_0 - P_+ \ ;$$
$$A_1 = -(1-\Delta)P_1 \ ,$$
$$G_1 = -(\mu + \mu'\Delta)P_1 \ ;$$
$$b = (A_1\nu - G_1)\frac{1-1/\epsilon}{\cos\theta} - \frac{P_1}{\epsilon}\beta + P_3 - P_4 \ ,$$
$$h = (-A_1 + G_1\nu')\frac{1-1/\epsilon}{\cos\theta} + \frac{P_1}{\epsilon}\beta\nu' + P_3\nu + P_4\nu' \ ;$$
$$\langle B_1\rangle = b + \psi P_- \ ,$$
$$\langle H_1\rangle = h + \psi P_+ \ ;$$
$$\langle|B_1|^2\rangle = |b|^2 + \tfrac{4}{3}|\psi P_-|^2 + \psi(b^*P_- - bP_-^*) ,$$
$$\langle|H_1|^2\rangle = |h|^2 + \tfrac{4}{3}|\psi P_+|^2 + \psi(h^*P_+ - hP_+^*) \ ;$$

$$\langle A_2\rangle = (1-\Delta)(P_3+\psi P_1) - P_1\beta\Delta \ ,$$

$$\langle G_2\rangle = (\mu+\mu'\Delta)(P_3+\psi P_1) - P_1\beta\Delta\mu' \ ;$$

$$\langle B_2\rangle = (\langle A_2\rangle\nu - \langle G_2\rangle)\left(1-\frac{1}{\epsilon}\right)\frac{\sin^2\theta}{2\cos\theta} + f\langle l_2\rangle - f'\langle l_2'\rangle + \tfrac{1}{2}(P_1-P_5) \ ,$$

$$\langle H_2\rangle = (-\langle A_2\rangle + \langle G_2\rangle\nu')\left(1-\frac{1}{\epsilon}\right)\frac{\sin^2\theta}{2\cos\theta} + f\langle l_2\rangle\nu + f'\langle l_2'\rangle\nu' + \tfrac{1}{2}(P_1\nu + P_5\nu') \ ;$$

$$f = 1/(\cos^2\theta - \beta^{-2}) \ , \qquad f' = 1/(\epsilon(\omega) - \sin^2\theta - \beta^{-2}) \ ,$$

$$P_0 = \exp[-i(\omega/c)(d/\beta)] \ , \qquad P_1 = f' - f \ ,$$

$$P_2 = (f'/\epsilon - f)\Delta \ , \qquad P_- = P_1 - P_2 \ ,$$

$$P_+ = P_1\nu + P_2\nu' \ , \qquad P_3 = 2(f'^2 - f^2)/\beta$$

$$P_4 = 2\left(\frac{f'^2}{\epsilon} - f^2\right)\frac{\Delta}{\beta} \ , \qquad P_5 = -(P_3+\psi P_1)\frac{\sin^2\theta}{\epsilon\cos\theta} \ ,$$

$$\psi = i(\omega/c)\tfrac{1}{2}d \ , \qquad \Delta = 1/\beta\cos\theta \ ,$$

$$\delta = (\epsilon - \sin^2\theta)^{1/2}/\cos\theta \ ,$$

$$\langle l_2\rangle = f\left(\tfrac{1}{2}\sin^2\theta + \frac{\psi\sin^2\theta}{\beta} + \frac{1}{\beta^2}\right) + \frac{2f^2\sin^2\theta}{\beta^2} + \tfrac{1}{2}\psi + \tfrac{2}{3}\psi^2,$$

$$\langle l_2'\rangle = f'\left(\tfrac{1}{2}\sin^2\theta + \frac{\psi\sin^2\theta}{\beta} + \frac{1}{\beta^2}\right) + \frac{2f'^2\sin^2\theta}{\beta^2} + \tfrac{1}{2}\psi + \tfrac{2}{3}\psi^2.$$

A detailed analysis of accurate experimental data in light of the present results will be given in the subsequent paper.

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Temperature Variation of the dc Josephson Current in Pb-Pb Tunnel Junctions*

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The temperature variation of the dc Josephson current $J_s(T)$ of Pb-Pb tunnel junctions has been investigated both experimentally and theoretically. The experimental data do not agree with the temperature dependence of the dc Josephson current derived by Ambegaokar and Baratoff for the case of two weak coupling superconductors. However, detailed numerical calculations of the temperature variation of the dc Josephson current for a Pb-Pb tunnel junction, which employ strong coupling superconductivity theory throughout, are in reasonably good agreement with the experimental measurements.

I. INTRODUCTION

The work to be described is a joint experimental and theoretical investigation of the temperature variation of the dc Josephson current for a Pb-Pb tunnel junction. The experimental work was carried out by the group at Waterloo, while the theoretical work was performed by the group at McMaster.

Previous experimental work on the temperature variation of the dc Josephson current has been interpreted as supporting the temperature dependence derived by Ambegaokar and Baratoff[1] for the case of two weak coupling superconductors. For example, both Fiske[2] for Sn-Sn and Pb-Sn Josephson junctions and Hauser[3] for Pb-Pb Josephson junctions concluded that the temperature dependence of the dc Josephson current was described by the Ambegaokar-Baratoff formula. However, while Fiske's data for an Sn-Sn Joseph-